A Numerical Thermoelastoplastic Solution of a Thick-Walled Tube

SHIH-CHI CHU*
Rock Island Arsenal, Rock Island, Ill.

An incremental theory is presented for solving the problem of elastoplastic thick-walled tubes subjected to transient thermal loading. The effective stress in a thick-walled tube subjected to thermal loading is shown by examples that it is not a monotonic function with respect to the radius, but is strongly dependent upon the history, gradient, and distribution of the temperature in the tube. The multiple elastic-plastic boundary problem of a thick-walled tube can be solved by use of the presented theory. Because temperature history is included in the analysis, the theory is particularly suitable for predicting stress and strain distributions, and locations of the elastic-plastic boundaries of a thick-walled tube subjected to transient-state thermal loading.

Nomenclature

$\sigma, \boldsymbol{\sigma} = \sigma/\sigma_0$	= normal stress and dimensionless normal stress, respectively
$\sigma_r, \sigma_\theta, \sigma_z$	= principal stresses
$\boldsymbol{\sigma}_r = \sigma_r/\sigma_0$, etc.	= dimensionless components of stress
$\varepsilon, \boldsymbol{\varepsilon} = \varepsilon/\varepsilon_0$	= normal strain and dimensionless normal strain, respectively
$\varepsilon_r, \varepsilon_\theta, \varepsilon_z$	= principal strains
$\boldsymbol{\varepsilon}_r = \varepsilon_r/\varepsilon_0$, etc.	= dimensionless components of strain
$\varepsilon^e, \varepsilon^p$	= elastic and inelastic normal strain, respectively
$\bar{\sigma}, \bar{\boldsymbol{\sigma}} = \bar{\sigma}/\sigma_0$	= effective stress and dimensionless effective stress, respectively
$\bar{\varepsilon}, \bar{\boldsymbol{\varepsilon}} = \bar{\varepsilon}/\varepsilon_0$	= effective strain and dimensionless effective strain, respectively
$r, \rho = r/a$	= radius and dimensionless radius, respectively
a, b	= inside and outside radius of a thick-walled tube
E	= Young's modulus
v	= Poisson's ratio
η	= strain-hardening factor for material
α	= coefficient of thermal expansion
$\sigma_0, \varepsilon_0 = \sigma_0/E$	= yielding stress and yielding strain in tension, respectively
S, e	= mean normal stress and mean normal strain, respectively
$\mathbf{S} = S/\sigma_0, \mathbf{e} = e/\varepsilon_0$	= dimensionless mean normal stress and dimensionless mean normal strain, respectively
e_r, e_θ, e_z	= the deviator components of strain in the direction of r, θ, and z, respectively
$\mathbf{e}_r = e_r/\varepsilon_0$, etc.	= dimensionless deviator components of strain
S_r, S_θ, S_z	= the deviator components of stress in the direction of r, θ, z, respectively
$\mathbf{S}_r = S_r/\sigma_0$, etc.	= dimensionless deviator components of stress
T	= temperature

Introduction

A LITERATURE survey indicates that the problems of elastoplastic thick-walled tubes have been solved by most investigators[1-13] based on tubes subjected to mechanical loadings, such as internal and external pressures and axial loads. Very little work has been done on the elastoplastic solutions for thick-walled tubes subjected to transient thermal loading. The effective stress in a thick-walled tube under pressure and axial loading has been proved to be a monotonic function with respect to the radius; hence, only one elastic-plastic boundary exists in the tube, if it exists. However, for the problem of an elastoplastic thick-walled tube subjected to thermal loading, the effective stress in the tube is not necessarily a monotonic function with respect to radius. The effective stress in the tube is strongly dependent upon the heat flow in the wall of a thick-walled tube. Therefore, the number of elastic-plastic boundaries in the wall of the tube is not necessarily restricted to be one, but may be two or more appearing at several places in the wall of a tube. Distribution, gradient, and history of the temperature in a tube are very important factors in determining the locations of the elastic-plastic boundaries. Therefore, this problem is more difficult than the problem of a thick-walled tube subjected only to pressure and axial loads.

On the basis of Tresca's yield criterion and its associated flow rule, Bland[14] presented a closed-form elastoplastic solution for a thick-walled tube made of linear work-hardening material including the effects of temperature gradients. On the basis of Bland's approach, the stress and strain at any state are dependent upon the instantaneous state of stress, strain, loading, and temperature, and not on how that stress, strain, loading, and temperature system is reached. By use of Prandtl-Reuss' incremental stress-strain rule, von Mises' yield criterion, and the compressibility and strain-hardening properties of a material, an efficient incremental theory has been developed for solving the problem of thermoelastoplastic thick-walled tubes.

Requirements of a Thermoelastoplastic Solution

The conditions that should be fulfilled by a satisfactory thermoelastoplastic solution of a thick-walled tube subjected to temperature gradients are as follows:

1) Strain compatibility condition

$$d\varepsilon_\theta/dr = (1/r)(\varepsilon_r - \varepsilon_\theta) \tag{1}$$

2) Equation of equilibrium

$$d\sigma_r/dr = (1/r)(\sigma_\theta - \sigma_r) \tag{2}$$

3) Boundary conditions

$$\sigma_r = 0 \quad \text{at} \quad r = a \quad \text{and} \quad b \tag{3}$$

4) Plane sections remain plane.

5) All stresses and strains are continuous across the elastic-plastic boundary. Stresses and strains that are continuous across the elastic-plastic interface can be obtained provided the same compressibility is assumed for stress-strain relations in the elastic region as for the elastic strain in the plastic region.

Received November 20, 1972; presented as Paper 73-256 at the AIAA 11th Aerospace Sciences Meeting, Washington, D.C., January 10–12, 1973; revision received August 1, 1973.

Index category: Thermal Stresses.

* Mechanical Engineer, Research Directorate, Gen. Thomas J. Rodman Laboratory.

6) Stress-strain relations: a) In an elastic region, the Duhamel-Neumann's stress-strain-temperature relations must be satisfied; b) In an inelastic region, the Prandtl-Reuss' incremental stress-strain rules should be satisfied.

7) Compressibility of Material. The total strain for a volume element that has been inelastically deformed consists of an elastic strain and an inelastic strain, thus

$$\varepsilon = \varepsilon^e + \varepsilon^p \tag{4}$$

in which ε^e and ε^p are elastic and inelastic strains, respectively. Experiments[15] have shown that the inelastic components of strain do not contribute to volume change. In other words, the volume change is always elastic, hence

$$Ede = (1-2v)ds \tag{5}$$

If the stresses and strains satisfy the equation above, the material is said to be compressible. A material is said to be incompressible if

$$de = 0 \tag{6}$$

For reduction of the complexity of the mathematical work, the elastic volume changes are usually neglected. In this investigation, however, the compressibility of the material will be considered.

8) The two most accepted yield criteria are the Tresca maximum shearing stress yield criterion and the von Mises' yield criterion. They are usually stated as

$$f = \operatorname*{Sup}_{i,j} |\sigma_i - \sigma_j| - \sigma_0 = 0$$

$$i, j = 1, 2, 3 \text{ (Tresca)} \tag{7}$$

and

$$g = \tfrac{1}{2}S_{ij}S_{ij} - \tfrac{1}{3}\sigma_0^2 = 0$$

$$i, j = 1, 2, 3 \text{ (von Mises)} \tag{8}$$

9) The most popular loading function for a given strain-hardening material may be written as $\bar{\sigma} = h(\bar{\varepsilon})$ for all combinations of stress where the effective stress $\bar{\sigma}$ and the effective strain $\bar{\varepsilon}$ are defined by

$$\bar{\sigma} = (\tfrac{3}{2})^{1/2}(S_{ij}S_{ij})^{1/2} \tag{9}$$

and

$$\bar{\varepsilon} = (\tfrac{2}{3})^{1/2}(e_{ij}e_{ij})^{1/2} \tag{10}$$

Theory

Assumptions

The development of an incremental theory for thick-walled tubes subjected to temperature gradients is based on the following assumptions: a) Deformation is small, and the boundary conditions at the surfaces during the flow are determined by the radii of the unstrained tube. b) Material is homogeneous and isotropic. c) Material is compressible. d) Material properties such as yield surface, modulus of elasticity, and Poisson's ratio are temperature-independent. However, this assumption is unnecessary if the relations of material properties and temperature are given. e) Prandtl-Reuss' incremental stress-strain rule is valid. f) von Mises' yield criterion is valid.

Incremental Theory for Thermoelastoplastic Thick-walled Tubes

The analysis and computation may be simplified by conversion of the stresses and strains to dimensionless stress and strain components as indicated in the Nomenclature.

In the development of a thermoelastoplastic solution of a thick-walled tube, a cylindrical coordinate system (r, θ, z) is used with the z axis coincident to the axis of the tube.

The cross section of a thick-walled tube is divided into n rings by dimensionless radii $\rho_1 = 1, \rho_2, \ldots, \rho_k = C, \ldots, \rho_{n+1} = b/a$ where $\rho_k = C$ is an elastic-plastic boundary.

The heat flow applied to the thick-walled tube will be considered step by step. For each increment of temperature $\Delta T(r)$, the finite-difference method is used to solve the following formulated problem.

At any point $\rho = \rho_i$ in the elastic region, the Duhamel-Neumann's stress-strain-temperature relations are assumed to be satisfied[14]

$$\Delta\varepsilon_r^e = \Delta\sigma_r - v\Delta\sigma_\theta - v\Delta\sigma_z + \alpha\Delta T/\varepsilon_0 \tag{11a}$$

$$\Delta\varepsilon_\theta^e = -v\Delta\sigma_r + \Delta\sigma_\theta - v\Delta\sigma_z + \alpha\Delta T/\varepsilon_0 \tag{11b}$$

$$\Delta\varepsilon_z^e = -v\Delta\sigma_r - v\Delta\sigma_\theta + \Delta\sigma_z + \alpha\Delta T/\varepsilon_0 \tag{11c}$$

Prandtl-Reuss' incremental stress-strain relations are assumed to be valid at any point in the plastic region

$$\Delta\varepsilon_r^p/S_r = \Delta\varepsilon_\theta^p/S_\theta = \Delta\varepsilon_z^p/S_z = \Delta\lambda \tag{12}$$

where $\Delta\lambda$ is a nonnegative constant which may vary with the position of the volume element in the tube and the stage of inelastic deformation. The total strain in an overstrained material consists of an elastic and an inelastic component. The elastic-strain components can be eliminated by use of Eqs. (11) and the definition of $S_r = \sigma_r - S = \frac{1}{3}(2\sigma_r - \sigma_\theta - \sigma_z)$, etc. Then Eq. (12) can be written as

$$\frac{\Delta\varepsilon_r - [\Delta\sigma_r - v(\Delta\sigma_\theta + \Delta\sigma_z)] - \alpha\Delta T/\varepsilon_0}{(2\sigma_r - \sigma_\theta - \sigma_z)} = \frac{\Delta\varepsilon_\theta - [\Delta\sigma_\theta - v(\Delta\sigma_z + \Delta\sigma_r)] - \alpha\Delta T/\varepsilon_0}{(2\sigma_\theta - \sigma_z - \sigma_r)} = \frac{\Delta\varepsilon_z - [\Delta\sigma_z - v(\Delta\sigma_r + \Delta\sigma_\theta)] - \alpha\Delta T/\varepsilon_0}{(2\sigma_z - \sigma_r - \sigma_\theta)} = 3\Delta\lambda \tag{13}$$

Two independent equations are derived as follows:

$$\begin{aligned} &-[(2\sigma_z - \sigma_r - \sigma_\theta) + v(2\sigma_r - \sigma_\theta - \sigma_z)]\Delta\sigma_r + v[(\sigma_z - \sigma_r - \sigma_\theta) - \\ &\quad (2\sigma_r - \sigma_\theta - \sigma_z)]\Delta\sigma_\theta + [(2\sigma_r - \sigma_\theta - \sigma_z) + \\ &\quad v(2\sigma_z - \sigma_r - \sigma_\theta)]\Delta\sigma_z + (2\sigma_z - \sigma_r - \sigma_\theta)\Delta\varepsilon_r = \\ &\quad (2\sigma_r - \sigma_\theta - \sigma_z)\Delta\varepsilon_z + (\alpha\Delta T/\varepsilon_0)[(2\sigma_z - \sigma_r - \sigma_\theta) - \\ &\quad (2\sigma_r - \sigma_\theta - \sigma_z)] \end{aligned} \tag{14}$$

and

$$\begin{aligned} &v[(2\sigma_z - \sigma_r - \sigma_\theta) - (2\sigma_\theta - \sigma_z - \sigma_r)]\Delta\sigma_r - [(2\sigma_z - \sigma_r - \sigma_\theta) + \\ &\quad v(2\sigma_\theta - \sigma_z - \sigma_r)]\Delta\sigma_\theta + [v(2\sigma_z - \sigma_r - \sigma_\theta) + \\ &\quad (2\sigma_\theta - \sigma_z - \sigma_r)]\Delta\sigma_z + (2\sigma_z - \sigma_r - \sigma_\theta)\Delta\varepsilon_\theta = \\ &\quad (2\sigma_\theta - \sigma_z - \sigma_r)\Delta\varepsilon_z + (\alpha\Delta T/\varepsilon_0)[(2\sigma_z - \sigma_r - \sigma_\theta) - \\ &\quad (2\sigma_\theta - \sigma_z - \sigma_r)] \end{aligned} \tag{15}$$

The condition under which inelastic deformation will occur in a load-carrying member is specified by the loading function. Hill[12] proposed a loading function for incremental theories that is valid for all states of stress. For a linear isotropic strain-hardening material, the loading function can be written.[13]

$$\begin{aligned} &\{(1/2\bar{\sigma})(2\sigma_r - \sigma_\theta - \sigma_z) + \bar{A}(1+v)[(2\varepsilon_r - \varepsilon_\theta - \varepsilon_z) - \\ &\quad (1+v)(2\sigma_r - \sigma_\theta - \sigma_z)]\}\Delta\sigma_r + \{(1/2\bar{\sigma})(2\sigma_\theta - \sigma_z - \sigma_r) + \\ &\quad \bar{A}(1+v)[(2\varepsilon_\theta - \varepsilon_z - \varepsilon_r) - (1+v)(2\sigma_\theta - \sigma_z - \sigma_r)]\}\Delta\sigma_\theta + \\ &\quad \{(1/2\bar{\sigma})(2\sigma_z - \sigma_r - \sigma_\theta) + \bar{A}(1+v)[(2\varepsilon_z - \varepsilon_r - \varepsilon_\theta) - \\ &\quad (1+v)(2\sigma_z - \sigma_r - \sigma_\theta)]\}\Delta\sigma_z - \bar{A}[(2\varepsilon_\theta - \varepsilon_z - \varepsilon_r) - \\ &\quad (1+v)(2\sigma_\theta - \sigma_z - \sigma_r)]\Delta\varepsilon_\theta - \bar{A}(2\varepsilon_r - \varepsilon_\theta - \varepsilon_z) - \\ &\quad (1+v)(2\sigma_r - \sigma_\theta - \sigma_z)\Delta\varepsilon_r = \bar{A}[(2\varepsilon_z - \varepsilon_r - \varepsilon_\theta) - \\ &\quad (1+v)(2\sigma_z - \sigma_r - \sigma_\theta)]\Delta\varepsilon_z \end{aligned} \tag{16}$$

in which

$$\bar{A} = \tfrac{2}{9}[\eta/(1-\eta)]1/\bar{\varepsilon}^p \tag{17}$$

$$\bar{\varepsilon}^p = [(2)^{1/2}/3][(\varepsilon_r^p - \varepsilon_\theta^p)^2 + (\varepsilon_\theta^p - \varepsilon_z^p)^2 + (\varepsilon_z^p - \varepsilon_r^p)^2]^{1/2} \tag{18}$$

and

$$\bar{\sigma} = [1/(2)^{1/2}][(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2]^{1/2} \tag{19}$$

The equation of equilibrium and the equation of compatibility are valid for both the elastic and the inelastic regions of a thick-walled tube. The finite-difference forms of these two equations in dimensionless stresses and strains, at any point ρ_i in the tube, are given by

$$(\rho_{i+1} - 2\rho_i)(\Delta\sigma_r)_i - (\rho_{i+1} - \rho_i)(\Delta\sigma_\theta)_i + \rho_i(\Delta\sigma_r)_{i+1} = (\rho_{i+1} - \rho_i)(\sigma_\theta - \sigma_r)_i - \rho_i[(\sigma_r)_{i+1} - (\sigma_r)_i] \tag{20}$$

for the equation of equilibrium, and

$$-(\rho_{i+1}-\rho_i)(\Delta\varepsilon_r)_i+(\rho_{i+1}-\rho_i)(\Delta\varepsilon_\theta)_i+\rho_i(\Delta\varepsilon_\theta)_{i+1} = (\rho_{i+1}-\rho_i)(\varepsilon_r-\varepsilon_\theta)_i-\rho_i[(\varepsilon_\theta)_{i+1}-(\varepsilon_\theta)_i] \quad (21)$$

for the equation of compatibility.

At each point $\rho = \rho_i$ in the wall of a tube, six incremental quantities $\Delta\sigma_r$, $\Delta\sigma_\theta$, $\Delta\sigma_z$, $\Delta\varepsilon_r$, $\Delta\varepsilon_\theta$, and $\Delta\varepsilon_z$ are present that must be determined for each step of temperature variation. Accounting for the fact that the axial strain ε_z is independent of ρ (for plane strain $\varepsilon_z = 0$), if the increment of axial strain $\Delta\varepsilon_z$ is specified ($\Delta\varepsilon_z = 0$ for plane strain condition) in each increment of temperature, then one finds only five incremental unknowns present at each point. Hence, a total of $5(n+1)$ unknowns exists that must be determined for each increment of temperature. The five equations listed previously can be formulated at each point (except $\rho = b/a$) either in the elastic region or in the plastic region. At the outer surface, $\rho = b/a$, of a thick-walled tube, some information concerning quantities in Eqs. (20) and (21) is unavailable. Hence, the total number of equations is $5(n+1)-2$. For solving $5(n+1)$ unknowns, two additional equations, resulting from the boundary conditions, are

$$(\Delta\sigma_r)_{\rho=1} = 0 \quad (22)$$

and

$$(\Delta\sigma_r)_{\rho=b/a} = 0 \quad (23)$$

Note that the numerical computation starts with given temperature distribution and temperature history. At the beginning of each variation of temperature, the distribution of stresses and strains is assumed to be known. For each step variation of temperature distribution, a system of $5(n+1)$ linear algebraic equations, with nonzero terms clustered about the main diagonal, will be obtained. This type of matrix is known as a band matrix and can be solved quite rapidly on a digital computer. In the computer program which was developed, the Gaussian elimination method was used to solve these equations.

Applications

Elastoplastic Solution of a Tube Subjected to Transient Temperature

The unsteady temperature field in a homogeneous, isotropic, thick-walled tube is here assumed to have axial and angular symmetry and, therefore, is dependent only upon time and radius distance. If the thermal properties are assumed to be independent of the temperature, the heat flow in a thick-walled tube is governed by the well-known Fourier heat conduction equation[16]

$$\partial^2 T/\partial\rho^2+(1/\rho)\partial T/\partial\rho = (1/\kappa)\partial T/\partial t \qquad 1<\rho<b/a, \quad t>0 \quad (24)$$

where $\kappa > 0$ denotes thermal diffusivity. In addition, the following initial and boundary conditions are specified for $T(\rho, t)$:

$$T(\rho,0) = f(\rho) \qquad 1<\rho<b/a, \quad t=0 \quad (25)$$

$$K\,\partial T/\partial\rho = -h_1(t)[T_g(t)-T_w]-\gamma F[T_g^4(t)-T_w^4], \quad \rho=1 \quad (26)$$

$$K\,\partial T/\partial\rho = -h_2(t)[T_w-T_a]-\gamma F[T_w^4-T_a^4], \quad \rho=b/a \quad (27)$$

where $T_g(t)$ and $T_a(t)$ are the temperature input functions at the bore and outer surfaces, T_w is the wall temperature, $h_1(t)$ and $h_2(t)$ are convection coefficients at bore and outer surfaces, respectively, γ is the Stefan-Boltzmann constant, and F is the interchange factor. If the uniform initial temperature distribution is assumed, then $f(\rho) = To$.

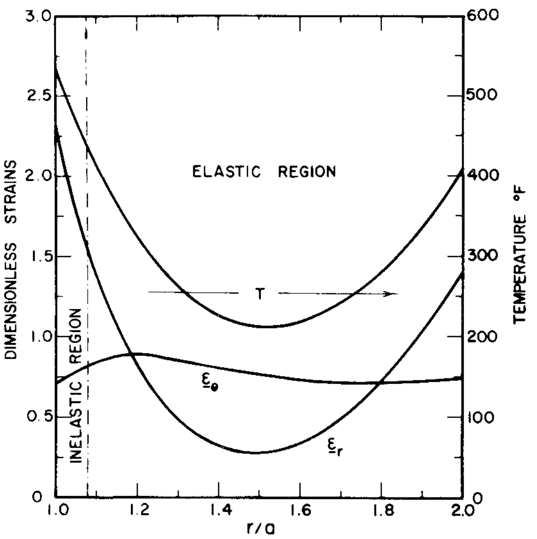

Fig. 2 Distribution of radial and circumferential strains of a partially yielded thick-walled tube subjected to the given transient temperature distribution (t = 79.7456 sec).

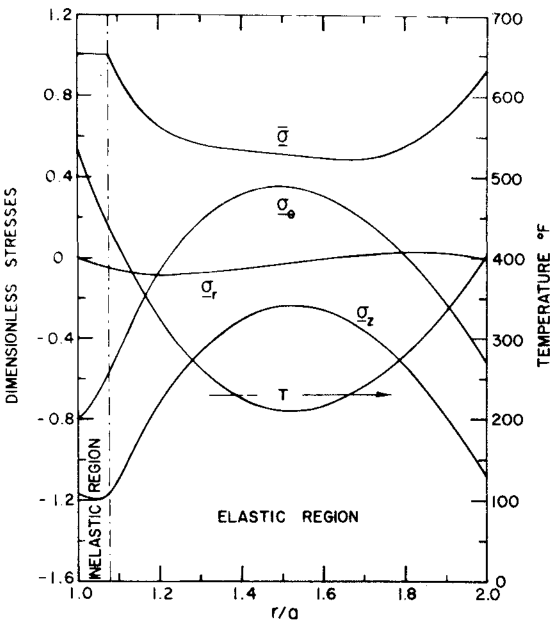

Fig. 1 Distribution of axial, radial, and circumferential stress components of a partially yielded thick-walled tube subjected to the given transient temperature distribution (t = 79.7456 sec).

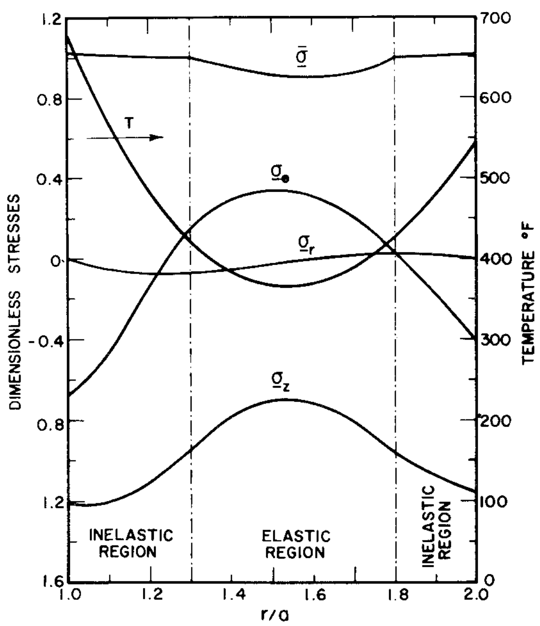

Fig. 3 Distribution of axial, radial, and circumferential stress components of a partially yielded thick-walled tube subjected to the given transient temperature distribution (t = 143.7990 sec).

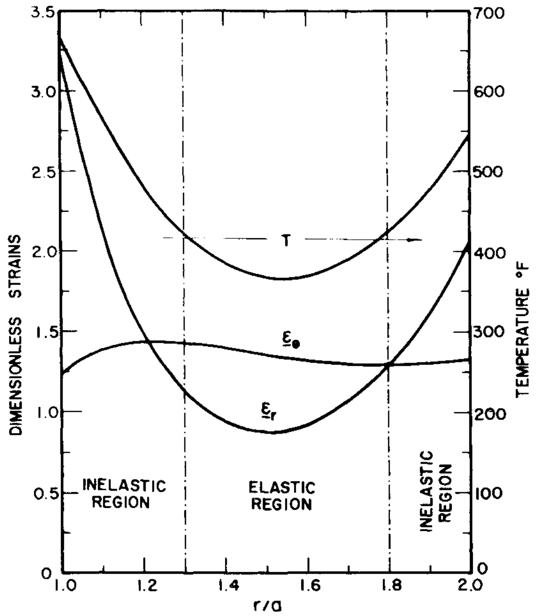

Fig. 4 Distribution of radial and circumferential strains of a partially yielded thick-walled tube subjected to the given transient temperature distribution (t = 143.7990 sec).

For a numerical solution, the following values were assigned: $K = 219$ (Btu/ft-hr-F), C specific heat = 0.0915 (Btu/lb-F), $\kappa = 4.353$ (ft^2/hr), $\gamma = 0.1714 \times 10^{-8}$ (Btu/hr-ft^2-R), $F = 0.2$, $T_g = 3000$°F, $T_a = 2000$°F, $h_1 = 100$ Btu/ft^2-hr.F, $h_2 = 100$ Btu/ft-hr.F, $T_0 = 100$°F, $b/a = 2.0$, $\nu = 0.3$, and $\eta = 0.05$

The solution to the problem of this transient temperature can be readily obtained by use of a finite-difference technique. For each time increment Δt, the temperature variation for that particular time increment is computed first, then this temperature variation is used as input to compute all corresponding incremental stresses and strains by the method described in a previous section. The distribution of stresses, strains, and temperatures at several times are given in Figs. 1–4. Only one elastic-plastic boundary exists in the tube as shown in Figs. 1 and 2. However, as time progresses, the temperature of the tube is raised; then, two elastic-plastic boundaries appear in the wall of the tube as shown in Figs. 3 and 4.

Conclusions and Recommendations

By use of an incremental theory, the thermoelastoplastic solution of a thick-walled tube is no longer restricted to the steady-state temperature distribution, but is extended to the unsteady-state temperature field. The effective stress in a thick-walled tube subjected to thermal loading has been shown not to be a monotonic function with respect to radius, but is strongly dependent upon the history, gradient and distribution of the temperature in the tube. The problem of multiple elastic-plastic boundaries of a thick-walled tube can be solved by use of the present theory. With the method presented in this paper, incremental stresses and strains are directly used as variables for each variation of temperature distribution. The stresses and strains in all principal directions can be computed at the same time, hence the same degree of accuracy of stresses and strains can be achieved.

The results presented in this paper were obtained on the assumption that all material properties such as yield surface, modulus of elasticity, and Poisson's ratio, are considered temperature independent; however, this assumption can readily be removed if the actual relations of material properties and temperature are given.

For expansion of the application of the present theory to actual engineering design problems, the following recommendations are suggested. 1) The present theory should be extended to include the effect of cyclic thermal loading. 2) Experimental tests should be performed to obtain the actual relations of material properties and temperature so that the assumption of temperature-independent material property can be removed. 3) Since the isotropic-hardening rule was assumed to be valid in this investigation, the Bauchinger effect was not considered. For extension of the present analysis to evaluate the fatigue life of a tube, the kinematic strain-hardening rule should be used instead of the isotropic-hardening rule.

References

[1] Cook, G., "The Stresses in Thick-Walled Cylinders of Mild Steel Over-Strained by Internal Pressure," *Proceedings of the Institution of Mechanical Engineers*, Vol. 126, 1934, p. 407.

[2] Sokolovsky, W. W., *Theory of Plasticity* (in Russian and German), Moscow, 1946, Chap. III.

[3] Hill, R., Lee, E. H., and Tupper, S. J., "The Theory of Combined Plastic and Elastic Deformation with Particular Reference to a Thick Tube Under Internal Pressure," *Proceedings of the Royal Society of London*, Ser. A, Vol. 191, 1947, p. 278.

[4] MacGregor, C. V., Coffin, L. E., Jr., and Fisher, J. C., "Partially Plastic Thick-Walled Tubes," *Journal of Franklin Institute*, Vol. 245, 1948, pp. 135–158.

[5] Nadai, A., *Theory of Flow and Fracture of Solids*, 2nd ed., Vol. 1, 1950, p. 460.

[6] Hodge, P. C. and White, G. N., "A Quantitative Comparison of Flow and Deformation Theories of Plasticity," *Journal of Applied Mechanics*, Vol. 72, 1950, pp. 180–184.

[7] Allen, D. N. and Sopwith, D. G., "The Stresses and Strains in a Partly Plastic Thick Tube Under Internal Pressure and End Load," *Proceedings of the Royal Society of London*, Ser. A, Vol. 205, 1951, p. 69.

[8] Hill, R., Lee, E. H., and Tupper, S. J., "Plastic Flow in a Closed-End Tube With Internal Pressure," *Proceedings of First U.S. National Congress of Applied Mechanics*, 1951, pp. 561–567.

[9] Steele, M. C., "Partially Plastic Thick-Walled Cylinder Theory," *Transactions of the ASME: Journal of Applied Mechanics*, Vol. 74, 1952, pp. 133–140.

[10] Hannon, B. M. and Sidebottom, O. M., "Plastic Behavior of Open-End and Closed-End Thick-Walled Cylinders," ASME Paper 67-WA/PVP-8, New York, 1967.

[11] Smith, J. O. and Sidebottom, O. M., *Inelastic Behavior of Load-Carrying Members*, Wiley, New York, 1965.

[12] Hill, R., *The Mathematical Theory of Plasticity*, Oxford Univ. Press, Oxford, 1950.

[13] Chu, S. C., "A More Rational Approach to the Problem of an Elastoplastic Thick-Walled Cylinder," *Journal of the Franklin Institute*, Vol. 294, No. 1, July 1972, pp. 57–65.

[14] Bland, D. R., "Elastoplastic Thick-Walled Tubes of Working-Hardening Material Subjected to Internal and External Pressures and to Temperature Gradients," *Journal of the Mechanics and Physics of Solids*, 1956, Vol. 4, pp. 209–229.

[15] Hanson, D. and Wheeler, N. A., "The Deformation of Metals Under Prolonged Loading," Pt. I, "The Flow and Fracture of Aluminum," *Journal of the Institute of Metals*, 1931, p. 45.

[16] Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids*, 2nd ed., Oxford Univ. Press, Oxford, 1959.